Journal of Pure and Applied Algebra 215 (2011) 450–454

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

On bimodules determining stable equivalences[☆]

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ARTICLE INFO

Article history:
Received 3 September 2009
Received in revised form 22 March 2010
Available online 8 May 2010
Communicated by I. Reiten

Dedicated to Andrzej Skowroński on the occasion of his sixtieth birthday

MSC: 16D20; 16G20

ABSTRACT

In the paper one shows that for two indecomposable non-simple self-injective algebras over a field K we have that if the functor $-\otimes_A N_B : \mathrm{mod}(A) \to \mathrm{mod}(B)$ induces a stable equivalence $\underline{\mathrm{mod}}(A) \to \underline{\mathrm{mod}}(B)$ then the bimodule ${}_AN_B$ is contained in the frame of a connected component in the Auslander–Reiten quiver $\Gamma_{A\otimes_K B^{op}}$.

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1. Introduction

Let K be a field. For a finite dimensional associative K-algebra A with an identity element 1 we shall denote by $\mathrm{mod}(A)$ the category of the finite dimensional left A-modules. Furthermore, its stable category $\underline{\mathrm{mod}}(A)$ modulo projectives is the quotient category $\mathrm{mod}(A)/\mathcal{P}$ whose objects are the same as those of $\mathrm{mod}(A)$, but the K-vector space $\underline{\mathrm{Hom}}_A(M, N)$ of morphisms from M to N in $\underline{\mathrm{mod}}(A)$ is defined to be the quotient vector space $\underline{\mathrm{Hom}}_A(M, N) = \mathrm{Hom}_A(M, N)/\mathcal{P}(M, N)$ of $\mathrm{Hom}_A(M, N)$, where $\mathcal{P}(M, N)$ consists of all homomorphisms in $\mathrm{Hom}_A(M, N)$ that factor through a projective A-module.

Following Brué (see [4]) two K-algebras A, B are said to be *stably equivalent of Morita type* if there are bimodules ${}_AN_B$, ${}_BM_A$ that satisfy the following conditions

(1) ${}_AN_B$, ${}_BM_A$ are projective as left modules and as right modules.
(2) ${}_AN_B \otimes_B M_A \cong A \oplus \Pi$ as A-bimodules for some projective A-bimodule Π.
(3) ${}_BM_A \otimes_A N_B \cong B \oplus \Pi'$ as B-bimodules for some projective B-bimodule Π'.

It is well known that if finite dimensional K-algebras A and B are stably equivalent of Morita type then their stable module categories are equivalent. Furthermore, the functor $-\otimes_A N : \mathrm{mod}(A) \longrightarrow \mathrm{mod}(B)$ induces an equivalence $\underline{\mathrm{mod}}(A) \longrightarrow \underline{\mathrm{mod}}(B)$. Therefore we shall say that ${}_AN_B$ is a bimodule that *determines a stable equivalence of A and B.*

In the representation theory of finite dimensional K-algebras stable equivalences of Morita type are of particular relevance. Their substantial role appears in the representation theory of finite groups (see [4,5,9]). Rickard observed that if two self-injective K-algebras are derived equivalent then they are stably equivalent of Morita type [8].

Our goal in this note is to look at stable equivalences of Morita type via bimodules that determine such equivalences. We are going to study some properties of these bimodules from the Auslander–Reiten theory point of view. The main result proved below indicates the positions of the bimodules in Auslander–Reiten quivers of the tensor product algebras. In fact they are contained in frames of components.

In order to state the main result of this note we should remind an invariant of indecomposable modules that was introduced by Auslander and Reiten in [3]. Let M be an indecomposable left A-module that is neither projective nor injective.

☆ Supported by the research grant No. N N201 269135 of the Polish Ministry of Science and Higher Education.
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doi:10.1016/j.jpaa.2010.04.029

Let $0 \to U \to V \to M \to 0$, $0 \to M \to X \to Y \to 0$ be Auslander–Reiten sequences in $\mathrm{mod}(A)$ (see [1,2]). Suppose that we have decompositions $V \cong V_1 \oplus \ldots \oplus V_n$, $X \cong X_1 \oplus \ldots \oplus X_m$ onto direct sums of indecomposable A-modules. Then

$$\alpha_A(M) = \max\{n, m\}.$$

The main result of this note is the following

Theorem 1.1. *Let A, B be two indecomposable non-simple self-injective finite dimensional K-algebras that are stably equivalent of Morita type. If ${}_AN_B$ is a bimodule that determines a stable equivalence of A and B then $\alpha_{A\otimes_K B^{op}}({}_AN_B) = 1$.*

2. Preliminaries on left–right projective bimodules

In what follows we fix two finite dimensional indecomposable self-injective K-algebras that are non-simple. We are interested in the left–right projective A-B-bimodules that will be interpreted as left $A \otimes_K B^{op}$-modules from the full subcategory $\mathrm{lrp}(A \otimes_K B^{op})$ in $\mathrm{mod}(A \otimes_K B^{op})$ formed by the modules that are projective as left A-modules and as right B-modules. Furthermore, we shall denote by $\underline{\mathrm{mod}}(A \otimes_K B^{op})$ (resp., $\underline{\mathrm{lrp}}(A \otimes_K B^{op})$) the quotient category $\mathrm{mod}(A \otimes_K B^{op})/\mathcal{P}$ (resp., $\mathrm{lrp}(A \otimes_K B^{op})/\mathcal{P}$), where $\mathcal{P}$ is the two-sided ideal in $\mathrm{mod}(A \otimes_K B^{op})$ (resp., $\mathrm{lrp}(A \otimes_K B^{op})$) consisting of the morphisms that factorize through projective $A \otimes_K B^{op}$-modules. We shall denote by $\underline{\mathrm{Hom}}_{A\otimes_K B^{op}}(X, Y)$ the morphism space between X and Y in $\underline{\mathrm{mod}}(A \otimes_K B^{op})$ that is the quotient space $\mathrm{Hom}_{A\otimes_K B^{op}}(X, Y)/\mathcal{P}(X, Y)$. Moreover, for every $f \in \mathrm{Hom}_{A\otimes_K B^{op}}(X, Y)$ we shall denote by $\underline{f}$ its coset modulo $\mathcal{P}(X, Y)$.

We shall use frequently the following facts proved in [6]. For any indecomposable left–right projective A-bimodule X that is not projective we have $\tau_{A^e}(X) \cong X \otimes_A \tau_{A^e}(A)$, $\tau^{-1}_{A^e}(X) \cong X \otimes_A \tau^{-1}_{A^e}(A)$ in $\underline{\mathrm{mod}}(A^e)$, where $A^e = A \otimes_K A^{op}$ is the enveloping algebra of A and τ_C is the Auslander–Reiten translation for an algebra C whose inverse is τ_C^{-1}. Furthermore, we have $\tau_{A\otimes_K B^{op}}(Y) \cong \tau_{A^e}(A) \otimes_A Y \cong Y \otimes_B \tau_{B^e}(B)$, $\tau^{-1}_{A\otimes_K B^{op}}(Y) \cong \tau^{-1}_{A^e}(A) \otimes_A Y \cong Y \otimes_B \tau^{-1}_{B^e}(B)$ in $\underline{\mathrm{mod}}(A \otimes_K B^{op})$ for any indecomposable left–right projective A-B-bimodule Y that is not projective. Moreover, we know from [7, Proposition 2.2] that if ${}_AN_B$, ${}_BM_A$ define a stable equivalence of Morita type between A and B then the following conditions are satisfied:

(1) There is an equivalence $F : \underline{\mathrm{lrp}}(A^e) \to \underline{\mathrm{lrp}}(B^e)$.

(2) There is an equivalence $F_1 : \underline{\mathrm{lrp}}(A^e) \to \underline{\mathrm{lrp}}(A \otimes_K B^{op})$.

(3) There is an equivalence $F_2 : \underline{\mathrm{lrp}}(A \otimes_K B^{op}) \to \underline{\mathrm{lrp}}(B^e)$.

Furthermore, $F = F_2 \circ F_1$ and F is induced by $M \otimes_A - \otimes_A N$, F_1 is induced by $- \otimes_A N$ and F_2 is induced by $M \otimes_A -$.

Lemma 2.1. *If a bimodule ${}_AN_B$ (resp., ${}_BM_A$) determines a stable equivalence of A and B (resp., B and A) then there are the following K-algebra isomorphisms:*

(1) $\underline{\mathrm{End}}_{A^e}(A) \cong \underline{\mathrm{End}}_{A\otimes_K B^{op}}({}_AN_B)$.
(2) $\underline{\mathrm{End}}_{B^e}(B) \cong \underline{\mathrm{End}}_{B\otimes_K A^{op}}({}_BM_A)$.
(3) $\underline{\mathrm{End}}_{A^e}(A) \cong \underline{\mathrm{End}}_{B^e}(B)$.

Proof. In order to prove (1) let us consider a homomorphism $\varphi : \underline{\mathrm{End}}_{A^e}(A) \to \underline{\mathrm{End}}_{A\otimes_K B^{op}}({}_AN_B)$ given by the formula $\varphi(\underline{f}) = \underline{f \otimes \mathrm{id}_N}$. Since $F_1 : \underline{\mathrm{lrp}}(A^e) \to \underline{\mathrm{lrp}}(A \otimes_K B^{op})$ is an equivalence that is induced by the functor $- \otimes_A N_B$, hence φ is a K-algebra isomorphism.

Similarly one can prove condition (2).

In order to prove (3), one can use an isomorphism $\psi : \underline{\mathrm{End}}_{A^e}(A) \to \underline{\mathrm{End}}_{B^e}(B)$ given by the formula $\psi(\underline{f}) = \underline{\mathrm{id}_M \otimes_A f \otimes_A \mathrm{id}_N}$. □

Lemma 2.2. *Let A be a K-algebra. For an indecomposable nonprojective left A-module V and a left A-module U we have that $f : U \to V$ is a split epimorphism if and only if $\underline{f}\,\underline{t} = \underline{\mathrm{id}_V}$ for some $t : V \to U$.*

Proof. If f is a split epimorphism then there is $g : V \to U$ with $fg = \mathrm{id}_V$. Thus $\underline{f}\,\underline{g} = \underline{\mathrm{id}_V}$.

Now suppose that $\underline{f}\,\underline{t} = \underline{\mathrm{id}_V}$ for some $t : V \to U$. Then $ft - \mathrm{id}_V = p$ for some $p : V \to V$ with $\underline{p} = 0$. Since V is indecomposable nonprojective, there is a positive integer n with $p^n = 0$. Consider $g = (t - tp + tp^2 - tp^3 + \ldots + (-1)^{n-1}tp^{n-1}) = t(\mathrm{id}_V - p + p^2 - p^3 + \ldots)$. Then we get $fg = ft(\mathrm{id}_V - p + p^2 - p^3 + \ldots) = (\mathrm{id}_V + p)(\mathrm{id}_V - p + p^2 - \ldots) = \mathrm{id}_V$, and so f is a split epimorphism. □

Lemma 2.3. *Let A be a K-algebra. Let V be an indecomposable nonprojective left A-module. If $f : U \to V$ is an epimorphism in $\mathrm{mod}(A)$ and $g : X \to V$ is not a split epimorphism then the following condition holds:*
there exists $h : X \to U$ with $g = fh$ if and only if there exists $t : X \to U$ with $\underline{g} = \underline{f}\,\underline{t}$.

Proof. Suppose that there exists $h : X \to U$ with $g = fh$. Then $\underline{g} = \underline{f}\,\underline{h}$ obviously.

Now suppose that there exists $t : X \to U$ with $\underline{g} = \underline{f}\,\underline{t}$. Then $g - ft = g_1$ and $\underline{g_1} = 0$. Thus $g_1 = r'r''$ for some $r' : P \to V$, $r'' : X \to P$, where P is projective. Since f is an epimorphism and P is projective we get that there exists $t' : P \to U$ with $ft' = r'$. Hence we have $g_1 = r'r'' = ft'r'' = ft_1$ for $t_1 = t'r''$. Thus $g - ft = g_1 = ft_1$. Consequently, $g = ft + ft_1 = f(t + t_1)$ which finishes the proof. □

Proposition 2.4. *Let A, B be two self-injective indecomposable non-simple K-algebras. Let ${}_AN_B$ determines a stable equivalence of A and B. If*

$$0 \to \tau_{A^e}(Z) \xrightarrow{f} W \xrightarrow{g} Z \to 0$$

is an Auslander–Reiten sequence in $\mathrm{lrp}(A^e)$ *then*

$$0 \to \tau_{A^e}(Z) \otimes_A N_B \xrightarrow{f \otimes \mathrm{id}_N} W \otimes_A N_B \xrightarrow{g \otimes \mathrm{id}_N} Z \otimes_A N_B \to 0$$

is a sum of an Auslander–Reiten sequence in $\mathrm{lrp}(A \otimes_K B^{op})$ *and a sequence*

$$0 \to P \to P \oplus Q \to Q \to 0$$

where P, Q are projective $A \otimes_K B^{op}$-modules.

Proof. First notice that the functor $- \otimes_A N_B$ is exact, because ${}_AN_B$ is a left flat A-module. Thus the sequence

$$0 \to \tau_{A^e}(Z) \otimes_A N_B \xrightarrow{f \otimes \mathrm{id}_N} W \otimes_A N_B \xrightarrow{g \otimes \mathrm{id}_N} Z \otimes_A N_B \to 0$$

is exact.

Suppose that $Z \otimes_A N_B$ has the maximal projective direct summand Q and $\tau_{A^e}(Z) \otimes_A N_B$ has the maximal projective direct summand P. Then, by self-injectivity of the algebra $A \otimes_K B^{op}$, we can split our sequence onto the short exact sequence $0 \to P \to P \oplus Q \to Q \to 0$ and a short exact sequence $0 \to U \to L \to V \to 0$ in $\mathrm{lrp}(A \otimes_K B^{op})$. The last sequence is not splittable by Lemma 2.2, because the sequence $0 \to \tau_{A^e}(Z) \xrightarrow{f} W \xrightarrow{g} Z \to 0$ does not split.

Moreover, we have $\tau_{A^e}(Z) \otimes_A N_B \cong \tau_{A^e}(A) \otimes_A Z \otimes_A N_B \cong \tau_{A \otimes_K B^{op}}(Z \otimes_A N_B)$ in $\underline{\mathrm{lrp}}(A \otimes_K B^{op})$. Now in our considerations we shall assume that $P = Q = 0$ for simplicity.

Suppose that $h : H \to Z \otimes_A N_B$ is not a splittable epimorphism. Then (under the assumption that $H \in \mathrm{lrp}(A \otimes_K B^{op})$) there is $r : H \to W \otimes_A N_B$ such that $h = (g \otimes \mathrm{id}_N)r$ if and only if $\underline{h} = (\underline{g \otimes \mathrm{id}_N})\underline{r}$ by Lemma 2.3. Since $H \in \underline{\mathrm{lrp}}(A \otimes_K B^{op})$, there is $\underline{h'} \in \underline{\mathrm{Hom}}_{A^e}(H \otimes_B M_A, Z)$ such that $\underline{h' \otimes \mathrm{id}_N} = \underline{h}$. Further we deduce from Lemma 2.2 that h' is not a splittable epimorphism. Then indeed there is $\underline{r'} \in \underline{\mathrm{Hom}}_{A^e}(H \otimes_B M_A, W)$ such that $\underline{h'} = \underline{g}\underline{r'}$. Putting $\underline{r} = \underline{r' \otimes \mathrm{id}_N}$ we obtain $\underline{h} = (\underline{g \otimes \mathrm{id}_N})(\underline{r' \otimes \mathrm{id}_N})$. Therefore for every homomorphism $h : H \to Z \otimes_A N_B$ that is not a splittable epimorphism the required property is satisfied provided that $H \in \mathrm{lrp}(A \otimes_K B^{op})$.

Now consider the case when $H \notin \mathrm{lrp}(A \otimes_K B^{op})$. We know from the Auslander–Reiten theory that there exists the Auslander–Reiten sequence

$$0 \to \tau_{A \otimes_K B^{op}}(Z \otimes_A N_B) \xrightarrow{p} X \xrightarrow{q} Z \otimes_A N_B \to 0.$$

Moreover, $\tau_{A \otimes_K B^{op}}(Z \otimes_A N_B) \in \mathrm{lrp}(A \otimes_K B^{op})$, hence $X \in \mathrm{lrp}(A \otimes_K B^{op})$. If $h : H \to Z \otimes_A N_B$ is not a splittable epimorphism then there is $t : H \to X$ such that $qt = h$. If we replace h by $q : X \to Z \otimes_A N_B$ with $\underline{q} = (\underline{g \otimes \mathrm{id}_N})\underline{r}$ for some $\underline{r} : X \to W \otimes_A N_B$ then $\underline{h} = \underline{q}\underline{t} = (\underline{g \otimes \mathrm{id}_N})\underline{r}\underline{t}$ and the required property holds for h which finishes the proof. □

3. Minimal objects

For a K-algebra C we fix a class $\mathcal{C}$ of objects in $\mathrm{mod}(C)$. We define a module $X \in \mathcal{C}$ to be *minimal in the class* $\mathcal{C}$ if there is no epimorphism $f : X \to Y$, $0 \neq Y \in \mathcal{C}$, that is not an isomorphism, and there is no monomorphism $g : Z \to X$, $0 \neq Z \in \mathcal{C}$, that is not an isomorphism.

Lemma 3.1. *If a module $X \in \mathcal{C}$ is minimal in the class $\mathcal{C}$ then it is indecomposable in the class $\mathcal{C}$.*

Proof. Assume to the contrary that $X \in \mathcal{C}$ has a decomposition $X \cong X_1 \oplus X_2$ in the class $\mathcal{C}$ (that means that $X_1, X_2 \in \mathcal{C}$), and X is minimal in the class $\mathcal{C}$. Then there is a splittable epimorphism $f : X \to X_1$ that is not an isomorphism in case of a non-trivial decomposition. Thus X is not minimal which contradicts to the assumption. Consequently, X is indecomposable in the class $\mathcal{C}$. □

Lemma 3.2. *Let A, B be two K-algebras. If $\mathcal{C} = \mathrm{lrp}(A \otimes_K B^{op})$ then every minimal object in the class $\mathcal{C}$ is indecomposable in* $\mathrm{mod}(A \otimes_K B^{op})$.

Proof. We infer by Lemma 3.1 that any minimal object X in the class $\mathcal{C}$ is indecomposable in this class. But if $X \cong X_1 \oplus X_2$ and $X \in \mathrm{lrp}(A \otimes_K B^{op})$, $X_1, X_2 \in \mathrm{mod}(A \otimes_K B^{op})$ then it is obvious that $X_1, X_2 \in \mathrm{lrp}(A \otimes_K B^{op})$ and the lemma is proved. □

Lemma 3.3. *If A is a self-injective K-algebra then the following conditions are equivalent:*

(1) *A is not minimal in the class* $\mathrm{lrp}(A^e)$.
(2) *There is an epimorphism of A-bimodules $f : A \to V$ that is not an isomorphism and* $0 \neq V \in \mathrm{lrp}(A^e)$.
(3) *There is a monomorphism of A-bimodules $g : U \to A$ that is not an isomorphism and* $0 \neq U \in \mathrm{lrp}(A^e)$.
(4) *There is a proper two-sided ideal $0 \neq I$ in the algebra A such that* $A/I \in \mathrm{lrp}(A^e)$.

Proof. We start the proof with showing that (2) is equivalent to (3). If there is an epimorphism $f : A \to V$ of A-bimodules that is not an epimorphism and $0 \neq V \in \mathrm{lrp}(A^e)$ then consider the identity embedding $g : \ker(f) \to A$. It is clear that $\ker(f) \neq 0$ and $\ker(f) \in \mathrm{lrp}(A^e)$. Moreover, g is not an isomorphism.

If there is a monomorphism $g : U \to A$ of A-bimodules that is not an isomorphism and $0 \neq U \in \mathrm{lrp}(A^e)$ then consider the natural epimorphism $f : A \to A/\mathrm{im}(g)$. Since g is not an isomorphism, $\mathrm{im}(g) \neq A$. Thus $A/\mathrm{im}(g) \neq 0$ and f is not an isomorphism. It is clear that $A/\mathrm{im}(g) \in \mathrm{lrp}(A^e)$.

Now we show that (1) is equivalent to (2). If A is not minimal in the class $\mathrm{lrp}(A^e)$ then there is an epimorphism $f : A \to V$ of A-bimodules that is not an isomorphism and $0 \neq V \in \mathrm{lrp}(A^e)$ or there is a monomorphism $g : U \to A$ of A-bimodules that is not an isomorphism and $0 \neq U \in \mathrm{lrp}(A^e)$. But in the second case we infer by the shown equivalence of (2) and (3) that there is a required epimorphism.

Now assume that there is an epimorphism $f : A \to V$ of A-bimodules that is not an isomorphism and $0 \neq V \in \mathrm{lrp}(A^e)$. Then we infer by definition that A is not minimal in the class $\mathrm{lrp}(A^e)$.

Finally we show that (2) is equivalent to (4). If there is a proper two-sided ideal $0 \neq I$ in A such that $A/I \in \mathrm{lrp}(A^e)$ then the natural epimorphism $f : A \to A/I$ satisfies the required conditions of (2). If there is an epimorphism $f : A \to V$ of A-bimodules that is not an isomorphism and $0 \neq V \in \mathrm{lrp}(A^e)$ then $\ker(f)$ is a proper two-sided ideal such that $A/\ker(f) \cong V \in \mathrm{lrp}(A^e)$. □

Lemma 3.4. *Let A, B be indecomposable non-simple self-injective K-algebras. Let A be minimal in the class* $\mathrm{lrp}(A^e)$*. If an indecomposable A-B-bimodule ${}_AN_B \in \underline{\mathrm{lrp}}(A \otimes_K B^{op})$ determines a stable equivalence of A and B then N is minimal in the class* $\mathrm{lrp}(A \otimes_K B^{op})$.

Proof. Assume that ${}_AN_B \in \underline{\mathrm{lrp}}(A \otimes_K B^{op})$ determines a stable equivalence of A and B and is indecomposable. Then we infer by [7] that the functor $- \otimes_A N$ induces an equivalence $\underline{\mathrm{lrp}}(A^e) \longrightarrow \underline{\mathrm{lrp}}(A \otimes_K B^{op})$. Suppose that ${}_AN_B$ is not minimal in the class $\mathrm{lrp}(A \otimes_K B^{op})$. Then there is an epimorphism $f : N \to X$ that is not an isomorphism and $0 \neq X \in \mathrm{lrp}(A \otimes_K B^{op})$ or there is a monomorphism $g : Y \to N$ that is not an isomorphism and $0 \neq Y \in \mathrm{lrp}(A \otimes_K B^{op})$. First consider the case of the epimorphism $f : N \to X$. Since f is an epimorphism, we know that $\underline{f} \neq 0$. Then there is a morphism $h : A \to V$ such that $V \otimes_A N \cong X$ in $\underline{\mathrm{lrp}}(A \otimes_K B^{op})$ and $\underline{h \otimes \mathrm{id}_N} = \underline{f}$, where $V \in \mathrm{lrp}(A^e)$. Thus $h \otimes \mathrm{id}_N = f + p$ for some p satisfying $\underline{p} = 0$. Without lost of generality we may assume that V is indecomposable nonprojective. Observe that $\mathrm{im}(p) \subset \mathrm{rad}(X)$. Indeed, if it is not the case we have $\underline{\pi p} \neq 0$ as an epimorphism for a projection $\pi : X \to S$ onto a simple module S. But $\underline{p} = 0$, and so $\underline{\pi}\,\underline{p} = 0$. Consequently, we obtain that $h \otimes \mathrm{id}_N$ is an epimorphism, because f is.

Now suppose that h is not an epimorphism. Then $h = h_2h_1$, where $h_1 : A \to \mathrm{im}(h)$ is an epimorphism and $h_2 : \mathrm{im}(h) \to V$ is the identity embedding. Since A is minimal in $\mathrm{lrp}(A^e)$, we infer by Lemma 3.3 that $\mathrm{im}(h)$ does not belong to $\mathrm{lrp}(A^e)$. Then $(h_2 \otimes \mathrm{id}_N)(h_1 \otimes \mathrm{id}_N) = f + p$ and $h_2 \otimes \mathrm{id}_N$ is a monomorphism that is not an isomorphism, because N is a left flat A-module, and $\mathrm{im}(h) \otimes_A N$ does not belong to $\mathrm{lrp}(A \otimes_K B^{op})$, since $\mathrm{im}(h) \in \mathrm{lrp}(A^e)$ otherwise. Therefore f cannot be an epimorphism. The obtained contradiction shows that h must be an epimorphism which contradicts to minimality of A in the class $\mathrm{lrp}(A^e)$. Consequently, there is no epimorphism $f : N \to X$ in $\mathrm{lrp}(A \otimes_K B^{op})$ such that $X \neq 0$ and f is not an isomorphism.

If there is a monomorphism $g : Y \to N$ that is not an isomorphism and $0 \neq Y \in \mathrm{lrp}(A \otimes_K B^{op})$ then $0 \neq N/\mathrm{im}(g) \in \mathrm{lrp}(A \otimes_K B^{op})$ and the natural epimorphism $f : N \to N/\mathrm{im}(g)$ is not an isomorphism. Applying the above arguments we obtain that there is not such an f. Hence there is not the above g. Consequently, N is minimal in the class $\mathrm{lrp}(A \otimes_K B^{op})$. □

Lemma 3.5. *Let A be a non-simple indecomposable self-injective K-algebra. If A is minimal in the class* $\mathrm{lrp}(A^e)$ *then* $\alpha_{A^e}(A) = 1$.

Proof. Assume that A is minimal in the class $\mathrm{lrp}(A^e)$ and suppose that $\alpha_{A^e}(a) > 1$. Then there is an Auslander–Reiten sequence $0 \to A \to W \to \tau_{A^e}^{-1}(A) \to 0$ such that $W \cong W_1 \oplus \ldots \oplus W_n$, $n \geq 2$, or there is an Auslander–Reiten sequence $0 \to \tau_{A^e}(A) \to E \to A \to 0$ such that $E \cong E_1 \oplus \ldots \oplus E_m$, $m \geq 2$. We shall consider the first case, because in the second one we proceed similarly.

Let

$$0 \to A \xrightarrow{\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}} W_1 \oplus \ldots \oplus W_n \xrightarrow{(p_1,\ldots,p_n)} \tau_{A^e}^{-1}(A) \to 0$$

be the Auslander–Reiten sequence in $\mathrm{mod}(A^e)$ and $n \geq 2$. Since A is minimal in the class $\mathrm{lrp}(A^e)$, every w_i is a monomorphism for $i = 1, \ldots, n$. Furthermore $\tau_{A^e}^{-1}(A)$ determines a stable equivalence of A and A. Hence we infer by Lemma 3.4 that $\tau_{A^e}^{-1}(A)$ is minimal in the class $\mathrm{lrp}(A^e)$. Therefore every $p_1, \ldots, p_n$ is an epimorphism. But in the case we have that $w = \begin{pmatrix} w_2 \\ \vdots \\ w_n \end{pmatrix} : A \to W_2 \oplus \ldots \oplus W_n$ is a monomorphism and $p = (p_2, \ldots, p_n) : W_2 \oplus \ldots \oplus W_n \to \tau_{A^e}^{-1}(A)$ is an epimorphism. Thus $\dim_K(A) < \dim_K(W_1)$ and $\dim_K(\tau_{A^e}^{-1}(A)) < \dim_K(W_2 \oplus \ldots \oplus W_n)$, which is impossible. Consequently, $n = 1$ and the lemma is proved. □

Proposition 3.6. *Let A, B be indecomposable non-simple self-injective K-algebras. If A is minimal in the class* $\mathrm{lrp}(A^e)$ *and an indecomposable $N \in \underline{\mathrm{lrp}}(A \otimes_K B^{op})$ determines a stable equivalence of A and B then $\alpha_{A\otimes_K B^{op}}(N) = 1$.*

Proof. Since $\tau_{A^e}(A)\otimes_A N$ and $\tau_{A^e}^{-1}(A)\otimes_A N$ give stable equivalences of A and B, then by Lemma 3.4 $\tau_{A^e}(A)\otimes_A N$ and $\tau_{A^e}^{-1}(A)\otimes_A N$ are minimal in $\mathrm{lrp}(A \otimes_K B^{op})$, so they are indecomposable. Therefore

$$\tau_{A\otimes_K B^{op}}(N) \cong \tau_{A^e}(A) \otimes_A N \qquad \tau^{-1}_{A\otimes_K B^{op}}(N) \cong \tau_{A^e}^{-1}(A) \otimes_A N.$$

Now the same arguments as in Lemma 3.5 prove the required result. □

4. Proof of the main result

We start this section with a result whose easy proof was given by S. Kasjan.

Proposition 4.1. *Let A be an indecomposable K-algebra. If $0 \neq I$ is an indecomposable A-subbimodule of A that is left–right projective then $I = A$.*

Proof. Assume that $0 \neq I$ is an indecomposable A-subbimodule of A that is left–right projective. Then $I = eA = Af$ for some idempotents e, f in A. We shall show that $e = f$. Indeed, $e = af$ and $f = eb$ for some $a, b \in A$. Then we have $f = eb = e^2b = afeb = af^2 = af = e$. Thus we have $eA = Ae = eAe$.

Now we shall show that $[e, a] = ea - ae = 0$ for every $a \in A$. Since $[e, a] \in eAe$, we have $e[e, a]e = e^2ae - eae^2 = 0$. Therefore $ea - ae = 0$ for every $a \in A$. Thus e is a central idempotent in A. Consequently, $e = 1$ and the required condition holds. □

Theorem 4.2. *Let A be an indecomposable K-algebra. If A is self-injective then the A-bimodule A is minimal in the class* $\mathrm{lrp}(A^e)$.

Proof. The result is an obvious consequence of Proposition 4.1 and Lemma 3.3. □

Proof of Theorem 1.1. We infer by Theorem 4.2 that A is minimal in the class $\mathrm{lrp}(A^e)$. Then we deduce from Proposition 3.6 that $\alpha_{A\otimes_K B^{op}}({}_AN_B) = 1$. □

References

[1] I. Assem, D. Simson, A. Skowroński, Elements of the representation theory of associative algebras, in: Vol. 1, Techniques of Representation Theory, in: London Math. Soc. Student Texts, vol. 65, Cambridge Univ. Press, Cambridge, 2006.
[2] M. Auslander, I. Reiten, Representation theory of Artin algebras III, Comm. Algebra 3 (1975) 269–310.
[3] M. Auslander, I. Reiten, Representation theory of Artin algebras IV, Comm. Algebra 5 (1977) 443–518.
[4] M. Broué, Equivalences of Blocks of Group Algebras, in: V. Dlab, L.L. Scott (Eds.), Finite Dimensional Algebras and Related Topics, in: NATO ASI SeriC, vol. 424, Kluwer Academic Press, Dodrecht, 1992, pp. 1–26.
[5] M. Linckelmann, Stable equivalences of Morita type for self-injective algebras and p-groups, Math. Z. 223 (2001) 243–255.
[6] Z. Pogorzały, A new invariant of stable equivalences of Morita type, Proc. Amer. Math. Soc. 131 (2003) 343–349.
[7] Z. Pogorzały, Left-right projective bimodules and stable equivalences of Morita type, Colloq. Math. 88 (2) (2001) 243–255.
[8] J. Rickard, Derived equivalences as derived functors, J. Lond. Math. Soc. (2) 43 (1991) 37–48.
[9] J. Rickard, Some recent advances in modular representation theory, in: CMS Conf. Proc., vol. 23, Amer. Math. Soc, 1998, pp. 157–178.